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TOPOSES AND INTUITIONISTIC THEORIES OF TYPES

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It is shown that by relaxing the conventional notion of translation and introducing 'comparisons' of translations as 2-arrows, intuitionistic theories of types form a 2-category which is equivalent to the 2-category of toposes with left exact functors and natural transformations. In this equivalence, translations preserving disjunctions (resp. the existential quantifiers, logical translations) correspond to functors preserving unions (resp. images, logical functors).

Introduction

It follows from the work of many authors (J. Bénabou, A. Boileau, M. Coste, A. Joyal, J. Lambek, M. Makkai, G.E. Reyes, P.J. Scott, ...) that there is an extremely close connection between the category of toposes with logical functors and the category of intuitionistic theories of types with 'translations' as morphisms. It is perhaps less well known that this connection may be turned into an 'equivalence', although there are suggestions in that direction in [3] (see especially Lemma 15.3, p. 204, Exercise 4 p. 205 and Exercise 4 p. 200). As formulated by Lambek and Scott, one should 'relax' the notion of translation. We do this as follows: a translation θ translates types into types, function symbols into functional formulas, relation symbols into formulas, but it should not be assumed that the domain of variation of a variable, say of type i, be sent to the whole domain of variation of type θi. In other terms and more generally a translation is a pair (θ, u) where θ 'translates' and u fixes for each type i a subdomain u_i of θi.

We note here that such a notion, although rarely conceptualized in detail, is already present in classical contexts (interpretations in set theory e.g. [5, pp. 260 ff.], relative interpretations in proofs of undecidability [6]).

One aim of this paper is to define a rather general concept of translation between theories of types. We will also sketch the proof that the 2-categories **Topos** of toposes with left exact functors as 1-arrows or with stronger funtors (e.g. logical) as 1-arrows are equivalent to the 2-categories **Th** of extensional intuitionistic theories of types having for 1-arrows translations or stronger translations (e.g.

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logical, i.e. preserving the whole extensional structure). The 2-arrows will be natural transformations on the one side and 'comparisons' of translations, their syntactic mimic on the other side. 'Equivalent' means here that we exhibit 2-functors

$$\Psi : \mathbf{Topos} \to \mathbf{Th} \quad \text{and} \quad \Phi : \mathbf{Th} \to \mathbf{Topos}$$

such that $\Phi\Psi(\mathscr{E})$ is equivalent to $\mathscr{E}$ and $\mathscr{T}$ is equivalent to $\Psi\Phi(\mathscr{T})$.

We will not insist on proofs or rigor in the definitions since many of the facts are well known (for a recent exposition, see again [3]). However, it will be necessary to recall some of these to fix our notation.

1. Intuitionistic theories of types with extensionality

1.1. A *language* $\mathscr{L}$ is given by the following data:

(a) A class Typ called class of *types*. A structure on this class constituted by (i) a constant $1 \in \mathrm{Typ}$; (ii) a binary law $\mathrm{Typ} \times \mathrm{Typ} \to \mathrm{Typ}$ which to i and j associates (i, j); (iii) a unary law $\mathrm{Typ} \to \mathrm{Typ}$ which to i associates Ω^i.

(b) For each type i, an infinite set Var_i of *variables* of type i.

(c) For each ordered pair (i, j) of types, a set $\mathrm{Op}_{i,j}$ of *operation symbols* of source i and target j. It is assumed that these sets contain projection symbols: π_{1,i_1,i_2} (or more simply π_1) $\in \mathrm{Op}_{(i_1,i_2),i_1}$ and $\pi_2 \in \mathrm{Op}_{(i_1,i_2),i_2}$.

(d) For each type i, a set Rel_i of *relation symbols* of signature i. It is assumed that these sets contain the equality and membership symbols: for each i, $=_i$ of signature (i, i) and $\in_i$ of signature (i, Ω^i).

(e) The logical symbols: conjunction $\wedge$ and collection or abstraction symbol $\{\,|\,\}$.

1.2. *Expressions* of $\mathscr{L}$ are defined inductively. Each expression is either a *term* or a *formula*. If it is a term, it has a *type*. The set of all formulas is denoted by Form and the set of all terms of type i is denoted by Term_i.

(a) If $x \in \mathrm{Var}_i$, then $x \in \mathrm{Term}_i$.

(b) $\phi \in \mathrm{Term}_1$.

(c) If $t_1 \in \mathrm{Term}_{i_1}$ and $t_2 \in \mathrm{Term}_{i_2}$, then $(t_1, t_2) \in \mathrm{Term}_{(i_1,i_2)}$.

(d) If $f \in \mathrm{Op}_{i,j}$ and $t \in \mathrm{Term}_i$, then $ft \in \mathrm{Term}_j$.

(e) If $r \in \mathrm{Rel}_i$ and $t \in \mathrm{Term}_i$, then $rt \in \mathrm{Form}$; in particular, if $t \in \mathrm{Term}_{\Omega^1}$, $\phi \in_1 t$ is a formula.

(f) If $\varphi, \psi \in \mathrm{Form}$, then $\varphi \wedge \psi \in \mathrm{Form}$.

(g) If $x \in \mathrm{Var}_i$ and $\varphi \in \mathrm{Form}$, then $\{x \mid \varphi\}$ is a term of type Ω^i.

Free variables are defined in the obvious way: $\{x \mid \varphi\}$ alone binds x. We denote by $V(E)$ the finite sequence $\boldsymbol{x} = (x_1, \dots, x_n)$ of free distinct variables of E and by $\tau(E)$ the corresponding sequence of their types.

Since we have projections and ordered pair operations, we may very often dispend with the consideration of sequences of variables. This is already present in our notation: $\varphi(x_1, x_2)$ ambigously denotes (i) either a formula with two free variables x_1

and x_2 (ii) or the result of the substitution of the term (x_1, x_2) for z in the formula $\varphi(z)$ (with $\tau(z) = (\tau(x_1), \tau(x_2))$; we even extend this ambiguity denoting by $\varphi(x_1, x_2)$ the formula $\varphi(z)$ itself).

The usual logical connectors are introduced as abbreviations ($x_1 \in \mathrm{Var}_1$ and $\xi \in \mathrm{Var}_{\Omega^1}$):

(a) $\varphi \leftrightarrow \psi$ stands for $\{x_1 \mid \varphi\} =_{\Omega^1} \{x_1 \mid \psi\}$,
(b) $\varphi \to \psi$ stands for $\varphi \leftrightarrow \varphi \wedge \psi$,
(c) T stands for $\phi =_1 \phi$,
(d) $\forall x\, \varphi$ stands for $\{x \mid \varphi\} = \{x \mid T\}$,
(e) $\perp$ stands for $\forall \xi\, (\phi \in_1 \xi\}$,
(f) $\varphi \vee \psi$ stands for $\forall \xi\, ((\varphi \to \phi \in \xi) \to (\psi \to \phi \in \xi) \to \phi \in \xi)$,
(g) $\exists x\, \varphi$ stands for $\forall \xi\, (\forall x\, (\varphi \to \phi \in \xi) \to \phi \in \xi)$,
(h) $\exists! x\, \varphi x$ stands for $\exists x\, \varphi x \wedge \forall y\, \forall y'\, (\varphi y \wedge \varphi y' \to y = y')$.

It is typographically inconvenient but very useful to have at hand *relativized quantifiers*: if α and φ are formulas, $\forall x^\alpha \varphi$, $\exists x^\alpha \varphi$, $\exists! x^\alpha \varphi x$ stand for $\forall x\, (\alpha \to \varphi)$, $\exists x\, (\alpha \wedge \varphi)$, $\exists x^\alpha\, \varphi x \wedge \forall y^\alpha\, \forall y'^\alpha\, (\varphi y \wedge \varphi y' \to y = y')$ respectively.

1.3. An (intuitionistic extensional) *theory of* (structured) *types* $\mathcal{T}$ in the language $\mathcal{L}$ is a class Γ of sentences of $\mathcal{L}$ together with the following axioms and rules of deduction (written in 'natural deduction'):

Axioms

(a) T,
(b) For each type i, $\forall x\, (x =_i x)$ (with $x \in \mathrm{Var}_i$),
(c) $\forall x_1\, (x_1 =_1 \phi)$ (with $x_1 \in \mathrm{Var}_1$),
(d) For each ordered pair (i_1, i_2) of types,

$$\forall x\, \forall y\, \pi_1(x, y) =_{i_1} x, \qquad \forall x\, \forall y\, \pi_2(x, y) =_{i_2} y$$

and $\forall z\, (z = (\pi_1 z, \pi_2 z))$ (with $x \in \mathrm{Var}_{i_1}, y \in \mathrm{Var}_{i_2}$ and $z \in \mathrm{Var}_{(i_1, i_2)}$).

Rules of deduction

(a) The usual rules of introduction and elimination of conjunction.
(b) Introduction and elimination of $\in$:

$$(\in I)\,\frac{[t/x]\varphi}{t \in \{x/\varphi\}}, \qquad (\in E)\,\frac{t \in (x/\varphi)}{[t/x]\varphi}$$

(the variable x and the term t having the same type).

(c) Introduction and elimination of $=$:

$$(= I)\,\frac{\begin{array}{cc} [x \in s] & [x \in t] \\ \vdots & \vdots \\ x \in t & x \in s \end{array}}{s = t}, \qquad (= E)\,\frac{s = t \quad [s/x]\varphi}{[t/x]\varphi}$$

(for $(=I)$, x is of type i, s and t have type Ω^i, $x \in s$ and $x \in t$ are discharged and x does not occur free in the other hypotheses; for $(=E)$, x, s and t have the same type).

1.4. Let us indicate some variants of presentation whose usefulness depends on the context.

(a) In some cases, it may be interesting to give a class of *sorts* and define types inductively: sorts are types, a finite sequence of types is a type and if i is a type, Ω^i is a type. Other version: sorts are types, and if $i_1, \ldots, i_n$ are types, then so is $\Omega^{i_1, \ldots, i_n}$.

(b) The basic set of logical symbols may also be modified. One can for example base the formalism on membership, implication, universal quantifier and collection symbol, thus omitting equality and conjunction. The rules of introduction and elimination for implication and universal quantifiers are the usual ones and those for membership and collection are those mentioned above. Conjunction $\varphi \wedge \psi$ is defined as an abbreviation for $\forall \xi\,((\varphi \rightarrow (\psi \rightarrow \phi \in \xi)) \rightarrow \phi \in \xi)$ (with $\xi \in \mathrm{Var}_{\Omega^1}$). Equality $s =_i t$ is defined 'from above' via Leibniz' principle: $\forall X\,(s \in_i X \leftrightarrow t \in_i X)$ with $X \in \mathrm{Var}_{\Omega^i}$. To obtain a theory which is deductively equivalent to the one we have described, it suffices to add for each i an axiom of extensionality:

$$\forall X\, \forall Y\, (\forall x\, (x \in_i X \leftrightarrow x \in_i Y) \rightarrow X =_{\Omega^i} Y)$$

(with x of type i, X and Y of type Ω^i).

This description is very close to the logician's description of (intuitionistic) theory of types (e.g. [4] and [2, p. 79 ff.]), the non-trivial difference is the plurality of types (or sorts) accepted here. It is particularly well-adapted for the techniques of proof theory.

(c) Even the collection symbol may be eliminated via axioms of comprehension:

$$\exists X\, \forall x\, (x \in_i X \leftrightarrow \varphi(x))$$

(with x of type i and X of type Ω^i).

1.5. We will use but do not recall here the concepts of *interpretation* and *model* of a theory in a topos.

2. Translations

2.1. Let $\mathcal{T} = (\mathcal{L}, \Gamma)$ and $\mathcal{T}' = (\mathcal{L}', \Gamma')$ be two theories. We describe hereafter a translation (θ, u) from $\mathcal{T}$ to $\mathcal{T}'$.

(a) θ maps the types of $\mathcal{L}$ into the types of $\mathcal{L}'$ and $\theta 1 = 1$, $\theta(i_1, i_2) = (\theta i_1, \theta i_2)$. It is not assumed in general that $\theta(\Omega^i) = \Omega^{\theta i}$.

(b) For each type i, u_i is a formula of $\mathcal{L}'$ in a variable x' of type θi. (Intuitively $u_i(x')$ describes in $\mathcal{L}'$ a universe which is contained in θi and is used to restrict the translation of variables of type i). It is assumed that u preserves part of the structure of types:

$$\Gamma' \vdash u_1(x_1') \leftrightarrow T$$

and

$$\Gamma' \vdash u_{(i_1,i_2)}(z') \leftrightarrow u_{i_1}(\pi_1 z') \wedge u_{i_2}(\pi_2 z').$$

(c) for each formula $\varphi(\boldsymbol{x})$, $\theta(\varphi(\boldsymbol{x}))$ is a formula in variables $\boldsymbol{x}'$ of type $\theta(\tau(\boldsymbol{x}))$. In full rigor, we should work with 'devariablized' formulas identifying $\varphi(\boldsymbol{x})$ and $[\boldsymbol{y}/\boldsymbol{x}]\varphi(\boldsymbol{x})$, whenever $\tau(\boldsymbol{y})=\tau(\boldsymbol{x})$ and substitution is legitimate. (In other words, the precise name of the free variables does not matter, only their types have to be taken into account). It is then assumed that $\theta(\varphi(\boldsymbol{x}))$ is equivalent in this 'substitutive' sense to $\theta([\boldsymbol{y}/\boldsymbol{x}]\varphi(\boldsymbol{x}))$. We will also write $\theta(\varphi)$, $\theta(\varphi(\boldsymbol{x}))$, $\theta(\varphi)(\boldsymbol{x}')$ or $\theta(\varphi(\boldsymbol{x}))(\boldsymbol{x}')$ according to the required degree of precision.

(d) One assumes also:

(α) A compatibility with 'true', conjunction, equalities and projections:

$$\Gamma' \vdash \theta(T);$$

$$\Gamma' \vdash u_i(x') \to (\theta(\varphi \wedge \psi)(x') \leftrightarrow (\theta\varphi \wedge \theta\psi)(x'))$$

(with $i=\tau(x)$ and $\tau(x')=\theta i$);

$$\Gamma' \vdash u_i(x') \wedge u_i(y') \to (\theta(x =_i y)(x', y') \leftrightarrow (x' =_{\theta i} y'));$$

$$\Gamma' \vdash u_{(i_1,i_2)}(z') \wedge u_{i_1}(x') \to (\theta(x =_{i_1} \pi_1 z)(x', z') \leftrightarrow (x' =_{\theta i_1} \pi_1 z'));$$

$$\Gamma' \vdash u_{(i_1,i_2)}(z') \wedge u_{i_2}(y') \to (\theta(y =_{i_2} \pi_2 z)(y', z') \leftrightarrow (y' =_{\theta i_2} \pi_2 z')).$$

(β) A preservation of proven equivalence: if $\Gamma \vdash \varphi(\boldsymbol{x}) \leftrightarrow \psi(\boldsymbol{x})$, then

$$\Gamma' \vdash u_i(\boldsymbol{x}') \to ((\theta\varphi)(\boldsymbol{x}') \leftrightarrow (\theta\psi)(\boldsymbol{x}'))$$

(with $\tau(\boldsymbol{x}) = i = (i_1, \ldots, i_n)$, $\tau(\boldsymbol{x}') = \theta i = (\theta i_1, \ldots, \theta i_n)$ and

$$u_i(\boldsymbol{x}') \equiv u_{i1}(x_1) \wedge \cdots \wedge u_{in}(x_n)).$$

Note that in particular, taking T for $\psi(\boldsymbol{x})$ one gets the preservation of theorems.

(γ) A preservation of proven functionality: if

$$\Gamma \vdash \forall_x^{\alpha}\, \exists!_y^{\beta}\, \varphi(x, y),$$

then

$$\Gamma' \vdash \forall_{x'}^{u_i \wedge \theta\alpha}\, \exists!_{y'}^{u_j \wedge \theta\beta}\, \theta\varphi(x, y)$$

(with $i=\tau(x)$ and $j=\tau(y)$).

(e) We agree to identify two translations (θ, u) and (θ^*, u^*) from $\mathscr{T}$ to $\mathscr{T}'$ if for all i,

$$\Gamma' \vdash u_i \leftrightarrow u_i^*$$

and for all formulas $\varphi(x)$,

$$\Gamma' \vdash u_i(x') \to (\theta\varphi(x') \leftrightarrow \theta^*\varphi(x')).$$

2.2. Given translations $(\theta, u): \mathscr{T} \to \mathscr{T}'$ and $(\theta', u'): \mathscr{T}' \to \mathscr{T}''$, their composition $(\theta'', u''): \mathscr{T} \to \mathscr{T}''$ is defined in the obvious way. For example, $u_i'' = u'_{\theta i} \wedge \theta'(u_i)$. Taking into account the identifications of translations, it is easy to verify that

Proposition 1. *Theories and translations form a category denoted by* **Th**.

It is interesting to consider subcategories of **Th** obtained by restricting the morphisms to translations compatible with certain logical connectors. For example, (θ, u) is compatible with $\vee$ if

$$\Gamma' \vdash u_i(x') \to (\theta(\varphi \vee \psi)(x') \leftrightarrow (\theta\varphi \vee \theta\psi)(x'))$$

and compatible with $\exists$ if

$$\Gamma' \vdash u_j(y') \to (\theta(\exists x\, \varphi(x, y))(y') \leftrightarrow \exists_{x'}^{u_i(x')}\, \theta\varphi(x', y')).$$

The strongest morphisms are *logical translations* which preserve every ingredient of the language. It suffices for that to ask that (θ, u) preserve exponentiation $(\theta(\Omega^i) = \Omega^{\theta i}$, $\Gamma' \vdash u_{\Omega^i}(X') \leftrightarrow \forall x' \in X'\, u_i(x'))$ and membership and collection:

$$\Gamma' \vdash u_{(i, \Omega^i)}(x', X') \to (\theta(x \in_i X)(x', X') \leftrightarrow x' \in_{\theta i} X'),$$

$$\Gamma' \vdash u_{(j, \Omega^j)}(y', X') \to (\theta(\{x \mid \varphi(x, y)\} = X)(y', X')$$
$$\leftrightarrow \{x' \mid u_i(x') \wedge (\theta\varphi)(x', y')\} = X')$$

(Note the relativization of x' to the universe u_i).

To describe a logical translation $(\theta, u): \mathscr{T} \to \mathscr{T}'$ it is not necessary to define θ on all formulas at once. Assume that θ and u have already been defined on types so as to preserve their 1, (,) and $\Omega^{(\ )}$-structure. Associate to each $f \in \mathrm{Op}_{i,j}$ a functional symbol $\theta f \in \mathrm{Op}_{\theta i, \theta j}$ or a formula $\theta f(x', y')$ and to each $r \in \mathrm{Rel}_i$ a relational symbol $\theta r \in \mathrm{Rel}_{\theta i}$ or a formula $\theta r(x', y')$ (with appropriate types) and extend θ to expressions inductively: e.g. define $\theta(x)(x', y')$ as $x' =_{\theta i} y'$ or $\theta(\{x \mid \varphi(x, y)\})(y', X')$ as $X' = \{x' \mid u_i(x') \wedge (\theta\varphi)(x', y')\}$. This will define a logical translation if for all f, θf is functional $(\Gamma' \vdash \forall_{x'}^{u_i(x')}\, \exists!_{y'}^{u_j(x')}\, \theta f(x', y'))$ and for all $\varphi \in \Gamma$, $\Gamma' \vdash \theta\varphi$.

2.3. Going back to general translations, we note here some preservation properties that may be derived essentially from the preservation of proven equivalence and proven functionality. In what follows, a notation like $u \to (\theta\varphi \to \theta\psi)$ stands for $u(\mathbf{x}') \to (\theta\varphi \to \theta\psi)$ where $\mathbf{x}'$ is a list of distinct variables containing those free in $\theta\varphi$ and $\theta\psi$.

Lemma 2. *Let* (θ, u) *be a translation from* $\mathscr{T}$ *to* $\mathscr{T}'$.

(a) *If* $\Gamma \vdash \varphi \to \psi$, *then* $\Gamma' \vdash u \to (\theta\varphi \to \theta\psi)$.

(b) $\Gamma' \vdash u_j \to (\theta(\forall x\, \varphi(x, y)) \to \forall_{x'}^{u_i}\, \theta\varphi)$, $\Gamma' \vdash u_j \to (\exists_{x'}^{u_i}\, \theta\varphi \to \theta(\exists x\, \varphi(x, y)))$ *(with* $\tau(x) = i$ *and* $\tau(y) = j$*).*

(c) $\Gamma' \vdash u \to (\theta\varphi \vee \theta\psi \to \theta(\varphi \vee \psi))$.

Proof. (a) comes from the preservation of proven equivalence since $(\varphi \to \psi)$ is logically equivalent to $(\varphi \wedge \psi \leftrightarrow \varphi)$ and θ preserves conjunction. The first part of (b) is derived as follows: $\Gamma \vdash \forall x\, \varphi(x,y) \to \varphi(x,y)$, hence by (a), $\Gamma' \vdash u_i \wedge u_j \to (\theta(\forall x\, \varphi(x,y)) \to \theta\varphi)$, hence $\Gamma' \vdash u_j \to (\theta(\forall x\, \varphi(x,y)) \to \forall_{x'}^{u_i} \theta\varphi)$ by obvious logical transformations. The second part of (b) is dual. (c) is consequence of (a).

Lemma 3. *If* $\Gamma \vdash \forall_y^{\beta} (\exists_x^{\alpha} \varphi(x,y) \to \exists! x^{\alpha} \varphi(x,y))$, *then*

$$\Gamma' \vdash \forall_{y'}^{u_j \wedge \theta\beta} (\theta(\exists x^{\alpha} \varphi)(y') \leftrightarrow \exists_{x'}^{u_i \wedge \theta\alpha} (\theta\varphi)(x',y')).$$

Proof. The implication from right to left comes from Lemma 2(b). The implication from left to right comes from the preservation of proven functionality since the hypothesis may be written as

$$\Gamma \vdash \forall_y^{\beta \wedge \exists x^{\alpha} \varphi} \exists! x^{\alpha} \varphi(xy).$$

2.4. It remains to define a notion of comparison of translations. Let

$$\mathscr{T} \underset{(\theta',u')}{\overset{(\theta,u)}{\rightrightarrows}} \mathscr{T}'$$

be two translations. A *comparison* $\gamma : (\theta,u) \Rightarrow (\theta',u')$ is given by fixing for each type i of $\mathscr{T}$ a formula $\gamma_i(x,y)$ of $\mathscr{L}'$ satisfying

(a) for each formula φ of $\mathscr{L}$ a functionality condition:

$$\Gamma' \vdash \forall_x^{u_i \wedge \theta\varphi} \exists!_y^{u_i' \wedge \theta'\varphi} \gamma_i(x,y);$$

(b) for formulas $\varphi(x)$, $\psi(y)$, $\xi(x,y)$ of $\mathscr{L}$ a naturality condition: if $\Gamma \vdash \forall_x^{\varphi} \exists!_y^{\psi} \xi(x,y)$, then

$$\begin{aligned} \Gamma' \vdash \forall_x^{u_i \wedge \theta\varphi} \forall_{y'}^{u_j' \wedge \theta'\psi} (&\exists_{x'}^{u_i' \wedge \theta'\varphi} (\gamma_i(x,x') \wedge \theta'\xi(x',y')) \\ &\leftrightarrow \exists_y^{u_j \wedge \theta\psi} (\theta\xi(x,y) \wedge \gamma_j(y,y'))) \end{aligned}$$

(The first condition essentially says that γ_i is a morphism from the 'structure' on u_i into the structure on u_j; the second essentially says that if ξ maps φ into ψ, then the 'diagram'

$$\begin{tikzcd} \theta\varphi \arrow[r, "\theta\xi"] \arrow[d, "\gamma_i"'] & \theta\psi \arrow[d, "\gamma_j"] \\ \theta'\varphi \arrow[r, "\theta'\xi"] & \theta'\psi \end{tikzcd}$$

commutes when appropriately restricted to u.).

It is easy to define the required vertical and horizontal composition to reinforce Proposition 1 into

Proposition 4. *Theories, translations and comparisons of translations form a 2-category.*

We note here that to define an isomorphic comparison γ between logical translations (θ, u) and (θ', u'), we can take advantage of their inductive description:

(a) For each type i, one gives a formula $\gamma_i(x,y)$ which is 'a bijection of u_i onto u_i''

$$(\Gamma' \vdash \forall_x^{u_i} \exists!_y^{u_j'} \gamma_i(x,y) \wedge \forall_y^{u_j'} \exists!_x^{u_i} \gamma_i(x,y))$$

and which 'preserves the structure of types'

$$(\Gamma' \vdash \gamma_1(\phi, \phi),$$

$$\Gamma' \vdash u_{(i_1,i_2)}(x) \wedge u'_{(i_1,i_2)}(y) \to (\gamma_{(i_1,i_2)}(x,y) \leftrightarrow \gamma_{i_1}(\pi_1 x, \pi_1 y) \wedge \gamma_{i_2}(\pi_2 x, \pi_2(y))$$

$$\Gamma' \vdash u_{\Omega^i}(X) \wedge u_{\Omega^i}(Y)$$

$$\to (\gamma_{\Omega^i}(X, Y) \leftrightarrow Y = \{y \mid u_i'(y) \wedge \exists_x^{u_i(x)} (x \in X \wedge \gamma_i(x,y))\}).$$

Clearly, the 'inverse' δ of γ is given by the formula $\delta(y,x) = \gamma(x,y)$.

(b) Suppose we can prove that for each $f \in \mathrm{Op}_{i,j}$, 'γ and δ commute with f', i.e.

$$\Gamma' \vdash \forall_x^{y_i} \forall_{y'}^{u_j'} (\exists_{x'}^{u_i'} (\gamma_i(x,x') \wedge \theta_f'(x',y')) \leftrightarrow \exists_y^{u_j} (\phi_f(x,y) \wedge \gamma_j(y,y')))$$

and similarly for δ.

(c) And finally suppose one can prove that for each $r \in \mathrm{Rel}_i$, 'γ maps θ_r onto θ_r'' and 'δ maps θ_r' onto θ_r', i.e.

$$\Gamma' \vdash \forall_{x'}^{u_i'} (\exists_x^{u_i} (\theta_r(x) \wedge \gamma_i(x,x')) \leftrightarrow \theta_r'(x'))$$

and similarly for δ.

This suffices to prove by induction on expressions that one has the analogue of (b) not only for operation symbols but for terms and the analogue of (c) not only for relation symbols but for formulas. It will then follow that γ and δ are inverse comparisons.

3. From theories to toposes

3.1. It is well known that to each (intuitionistic extensional) theory of (structured) types $\mathscr{T}$, one associates a topos $\mathscr{E}_{\mathscr{T}}$ which we call the syntactic topos associated with $\mathscr{T}$. (We read for the first time a description of $\mathscr{E}_{\mathscr{T}}$ in [1] and for the last time in [3] but there is obviously interference with work of other authors: J. Bénabou, A. Joyal, G. Reyes to quote but a few ones.)

To fix notations, but also with the hope to convince that those are manageable constructions, we recall without proof the essentials.

The objects of $\mathscr{E}_{\mathscr{T}}$ are equivalence classes of formulas in one variable, the equivalence being the one generated by deductive equivalence ($\Gamma \vdash \varphi(x) \leftrightarrow \varphi'(x)$) and substitutive equivalence evoked in 2.1(c). Of course, we continue to denote by the formula its equivalence class.

A morphism form $\varphi(x)$ to $\psi(y)$ (choosing x distinct from y) is an equivalence class of 'functional' formulas $\xi(x,y)(\Gamma \vdash \forall_x^{\varphi}\, \exists!_y^{\psi}\, \xi(x,y))$, the equivalence of $\xi(x,y)$ and $\xi'(x,y)$ being given by

$$\Gamma \vdash \forall_x^{\varphi}\, \forall_y^{\psi}\, (\xi(x,y) \leftrightarrow \xi'(x,y)).$$

The composition of $\varphi(x) \xrightarrow{\xi_1(x,y)} \psi(y) \xrightarrow{\xi_2(y,z)} \chi(z)$ is given by $\exists_y^{\psi}\, (\xi_1(x,y) \wedge \xi_2(y,z))$.

The identity on $\varphi(x)$ may be written as $\varphi(x) \xrightarrow{x=y} \varphi(y)$.

The object $\mathbf{1}$ of $\mathscr{E}_{\mathscr{T}}$ is $\phi =_1 \phi$.

The product of $\varphi(x)$ and $\psi(y)$ is given by

$$\varphi(x) \xleftarrow{\pi_1 z = x} \varphi(\pi_1 z) \wedge \psi(\pi_2 z) \xrightarrow{\pi_2 z = y} \psi(y)$$

(or written more loosely:

$$\varphi(x) \xleftarrow{x'=x} \varphi(x') \wedge \psi(y') \xrightarrow{y'=y} \psi(y)).$$

The equalizer of two arrows is given by

$$\chi(z) \xrightarrow{z=x} \varphi(x) \underset{\xi'(x,y)}{\overset{\xi(x,y)}{\rightrightarrows}} \psi(y).$$

where $\chi(z)$ is $\varphi(z) \wedge \exists_y^{\psi}\, (\xi(z,y) \wedge \xi'(z,y))$ or equivalently

$$\varphi(z) \wedge \forall_y^{\psi}\, (\xi(z,y) \leftrightarrow \xi'(z,y)).$$

The power object of the object $\varphi(x)$ is the object $P(\varphi(x))(X)$ given by $\forall x \in X\, \varphi(x)$, and to a binary relation

$$\varrho(z) \overset{\xi(z,y,x)}{\rightarrowtail} \psi(y) \wedge \varphi(x)$$

corresponds the morphism $\psi(y) \xrightarrow{\chi(y,X)} P(\varphi(x))(X)$ given by

$$\forall x\, (x \in X \leftrightarrow \exists_z^{\varrho}\, \xi(z,y,x)).$$

From all this, one derives the well-known proposition:

Proposition 5. *The category $\mathscr{E}_{\mathscr{T}}$ is a topos.*

As has been emphasized in [3, p. 200] one even has in $\mathscr{E}_{\mathscr{T}}$ a canonical choice of a mono representing a subobject:

Lemma 6. *If $\varphi(x) \xrightarrow{\xi(x,y)} \psi(y)$ is a mono in $\mathscr{E}_{\mathscr{T}}$, then there exists a unique object*

$\varrho(z)$ of $\mathscr{E}_{\mathscr{T}}$ such that $\varrho(z) \xrightarrow{z=y} \psi(y)$ be a monomorphism describing the same subobject as ξ.

Proof. Take $\varrho(z)$ given by the formula $\psi(z) \wedge \exists x\,(\varphi(x) \wedge \xi(x,z))$. The isomorphisms $\varphi(x) \rightleftarrows \varrho(z)$ are both given by $\xi(x,z)$.

3.2. The topos $\mathscr{E}_{\mathscr{T}}$ comes equipped with an interpretation $|\ \ |$ of $\mathscr{L}$: to the type i, one associates the object $x =_i x$, to $f \in \mathrm{Op}_{i,j}$ one associates the arrow $x =_i x \xrightarrow{f(x)=y} y =_i y$ and to $r \in \mathrm{Rel}_i$ one associates the subobject $ry \rightarrowtail^{y=x} x=x$. Under this interpretation, $\mathscr{E}_{\mathscr{T}}$ becomes a model of $\mathscr{T}$.

3.3. Let (θ, u) be a translation from $\mathscr{T}$ into $\mathscr{T}'$. We associate with it a functor $\Phi(\theta,u)$ of $\mathscr{E}_{\mathscr{T}}$ into $\mathscr{E}_{\mathscr{T}'}$: $\Phi(\theta,u)$ sends the object $\varphi(x)$ to the object $u_i(x') \wedge \theta(\varphi(x))(x')$ and sends the arrow $\varphi(x) \xrightarrow{\xi(x,y)} \psi(y)$ to $\theta\xi(x',y')$.

Proposition 7. *$\Phi(\theta,u)$ is a left-exact functor from $\mathscr{E}_{\mathscr{T}}$ to $\mathscr{E}_{\mathscr{T}'}$.*

Proof. (Sketch). $\Phi(\theta,u)$ is well defined since a translation preserves substitution and proven equivalence. $\Phi(\theta,u)(\xi(x,y))$ is an arrow since (θ,u) preserves proven functionality. As to functoriality, if $\varphi(x) \xrightarrow{\xi_1(x,y)} \psi(y) \xrightarrow{\xi_2(y,z)} \chi(z)$, then $\Phi(\theta,u)(\xi_2 \circ \xi_1) = \theta(\exists_y^{\psi}(\xi_1 \wedge \xi_2))$ and $\Phi(\theta,u)(\xi_2) \circ \Phi(\theta,u)(\xi_1) = \exists_{y'}^{u \wedge \theta\psi}(\theta\xi_1 \wedge \theta\xi_2)$; given the functionality of ξ_1, Lemma 3 furnishes the required equivalence of these formulas. Preservation of finite products is easy and preservation of equalizers amounts to another application of Lemma 3.

When completed by the adequate verifications, Proposition 7 essentially means that Φ is a functor from the category of theories to the category of toposes with left-exact functors. This functor induces also a functor from different subcategories of the category of theories and subcategories of the category of toposes with left-exact functors. We restrict our attention to the following result whose proof essentially uses the description of finite unions of subobjects, images and similar notions in the syntactic topos:

Proposition 8. (a) *If (θ,u) preserves disjunction, then $\Phi(\theta,u)$ preserves finite unions of subobjects.*

(b) *If (θ,u) preserves the existential quantifier, then $\Phi(\theta,u)$ preserves images.*

(c) *If (θ,u) is logical, then so is $\Phi(\theta,u)$.*

3.4. It remains to observe that the definition of a comparison $\gamma : (\theta,u) \Rightarrow (\theta',u')$ is exactly the assertion that the morphisms $\gamma_i(x,y)$ form a natural transformation from $\Phi(\theta,u)$ to $\Phi(\theta',u')$. Briefly stated, we arrive at

Proposition 9. *Φ is a 2-functor.*

4. From toposes to theories

4.1. To every topos $\mathscr{E}$ is associated a theory $\mathscr{T}_{\mathscr{E}}$. Its language $\mathscr{L}_{\mathscr{E}}$ is the internal language of $\mathscr{E}$ in the sense of J. Bénabou: the types and their structure are given by the objects of $\mathscr{E}$, $\mathrm{Op}_{i,j} = \mathscr{E}(i,j)$, $\mathrm{Rel}_i = \mathrm{Sub}(i)$, equalities, projections and membership relations are those of $\mathscr{E}$. The axioms of $\mathscr{T}_{\mathscr{E}}$ are the sentences valid in $\mathscr{E}$. There is an obvious interpretation $|\ \ |$ of $\mathscr{T}_{\mathscr{E}}$ in $\mathscr{E}$ under which $\mathscr{E}$ becomes model of $\mathscr{T}_{\mathscr{E}}$.

The language $\mathscr{L}_{\mathscr{E}}$ has a lot of operation and relation symbols: a consequence of this is a strong property of functional and relational completeness:

Lemma 10. (a) *The correspondence which to $f \in \mathscr{E}(i,j)$ associates the formula $\xi(x,y) \equiv (fx = y)$ is a bijection from $\mathscr{E}(i,j)$ onto the set of equivalence classes of functional everywhere defined formulas.*

(b) *The correspondence which to $(a \rightarrowtail i) \in \mathrm{Sub}(i)$ associates the formula ax (in x of type i) is a bijection from $\mathrm{Sub}(i)$ onto the set of equivalence classes of formulas (in x of type i). In $\mathscr{T}_{\mathscr{E}}$, every formula is equivalent to an atomic formula.*

Proof. The interpretation $|\varphi(x)|$ is a subobject of $|x| = i$, hence (b); (a) expresses the well-known topos-theoretic bijection between arrows and graphs.

Let $a = |\alpha(x)| \overset{m}{\rightarrowtail} i$ be a subobject in $\mathscr{E}$. Note that in $\mathscr{L}_{\mathscr{E}}$ there are variables x_i of type i and x_a of type a. The following lemma, whose proof is easy, compares formulas relativized to α and formulas with variables x_a:

Lemma 11. (a) $\mathscr{E} \models \forall x_i^{\alpha}\, \exists! x_a\, (x_i = m x_a)$.

(b) $\mathscr{E} \models \alpha(m x_a)$.

(c) $\mathscr{E} \models \forall x_i^{\alpha}\, \varphi(x_i) \leftrightarrow \forall x_a\, \varphi(m x_a)$.

(d) $\mathscr{E} \models \exists x_i^{\alpha}\, \varphi(x_i) \leftrightarrow \exists x_a\, \varphi(m x_a)$.

4.2. Let $F: \mathscr{E}_1 \to \mathscr{E}_2$ be a left-exact functor. We associate with F a translation $\Psi(F) = (\theta, u)$ from $\mathscr{T}_{\mathscr{E}_1}$ to $\mathscr{T}_{\mathscr{E}_2}$.

(a) For each type i_1 of $\mathscr{T}_{\mathscr{E}_1}$ (which is an object of $\mathscr{E}_1$), we let $\theta(i_1) = F(i_1)$ and $u_{i_1}(x')$ be $(x' =_{F(i_1)} x')$, i.e. 'true' in x'.

(b) If $\varphi(x)$ is a formula of $\mathscr{L}_{\mathscr{E}_1}$, $|\varphi| \overset{m}{\rightarrowtail} i$ is a subobject in $\mathscr{E}_1$ and F being left-exact transforms this in $F(|\varphi|) \overset{F(m)}{\rightarrowtail} F(i)$; we therefore define $\theta(\varphi(x))(x')$ to be the (atomic) formula $F(|\varphi|)x'$.

(c) Compatibility (2.1(d)(α)) with 'true', conjunction, equalities and projections are immediate consequences of the left-exactness of F.

(d) Preservation of proven equivalence is a consequence of Lemma 10(b) and preservation of functionality is a consequence of Lemmas 10(a) and 11. In this connection, we may note the following useful property:

Lemma 12. *If* $\mathscr{E}_1 \models \forall_x^{\varphi} \exists!_y^{\psi} \xi(x,y)$, *then, denoting by* $|[\xi]|_{\mathscr{E}_1} : |\varphi| \to |\psi|$ *the unique corresponding morphism, one has*

$$|[\theta(\xi)]|_{\mathscr{E}_2} = F(|[\xi]|_{\mathscr{E}_1}).$$

It is easy to check that

Proposition 13. $\Psi(F)$ *is a translation.*

Similarly, one verifies the functorial character of Ψ. This functor induces also functors from subcategories of toposes with left-exact functors to subcategories of theories with translations. Typically,

Proposition 14. (a) *If F preserves finite unions, then* $\Psi(F)$ *preserves disjunction.*
(b) *If F preserves images, then* $\Psi(F)$ *preserves the existential quantifier.*
(c) *If F is logical, then so is* $\Psi(F)$.

4.3. To a natural transformation $\alpha : F \Rightarrow G$ between left-exact functors F and G from $\mathscr{E}_1$ to $\mathscr{E}_2$, we associate a comparison $\Psi(\alpha) : \Psi(F) \Rightarrow \Psi(G)$.

Let i be a type of $\mathscr{T}_{\mathscr{E}_1}$, i.e. an object of $\mathscr{E}_1$; we let $\Psi(\alpha)_i(x', y')$ be the formula $(\alpha_i x' = y')$ where x' is of type $F_1(i)$ and y' of type $F_2(i)$.

Proposition 15. $\Psi(\alpha)$ *is a comparison of translations.*

Proof. Let us check for example the functionality condition. We have to show that

$$\mathscr{T}_{\mathscr{E}_2} \vdash \forall_{x'}^{\Psi(F_1)(\varphi)} \exists!_{y'}^{\Psi(F_2)(\varphi)} (\alpha_i x' = y'), \tag{$*$}$$

where x' is of type $F_1(i)$ and y' of type $F_2(i)$. Let $|\varphi| \overset{m}{\rightarrowtail} i$. By Lemma 10(a) applied to the arrow $\alpha_i \circ F_1(m)$,

$$\mathscr{E}_2 \models \forall \tilde{x}\, \exists! y_2\, (\alpha_i F_1(m)\tilde{x} = y_2)$$

where $\tilde{x}$ is of type $F_1(|\varphi|)$ and y_2 of type $F_2(i)$. Since α is a natural transformation, $\alpha_i \circ F_1(m) = F_2(m) \circ \alpha_{|\varphi|}$, i.e. expressed in $\mathscr{E}_2$:

$$\mathscr{E}_2 \models \forall \tilde{x}\, \exists! y_2\, \exists \tilde{y}\, (F_2(m)\tilde{y} = y_2 \wedge \alpha_{|\varphi|}\tilde{x} = \tilde{y})$$

where $\tilde{y}$ is of type $F_2(|\varphi|)$. But $F_2(m)$ is a mono, hence:

$$\mathscr{E}_2 \models \forall \tilde{x}\, \exists! \tilde{y}\, (\alpha_{|\varphi|}\tilde{x} = \tilde{y}),$$

$$\mathscr{E}_2 \models \forall \tilde{x}\, \exists! \tilde{y}\, \Psi(\alpha)_i(F_1(m)\tilde{x}, F_2(m)\tilde{y}),$$

and finally ($*$) by Lemma 10. Naturality conditions are checked by similar computations using Lemma 12.

We leave to the reader the ultimate verifications leading to

Proposition 16. *Ψ is a 2-functor.*

5. Equivalence of the 2-categories

5.1. Let us first consider the composition $\Phi \circ \Psi$:

$$\mathbf{Topos} \xrightarrow{\Psi} \mathbf{Th} \xrightarrow{\Phi} \mathbf{Topos}.$$

(a) Let $\mathscr{E}$ be a topos. The topos $\mathscr{E}' = \Phi(\Psi(\mathscr{E}))$ is equivalent to $\mathscr{E}'$. First, there is a functor $F : \mathscr{E} \to \mathscr{E}'$ which to $i \xrightarrow{f} j$ associates $(x =_i x) \xrightarrow{fx=y} (y =_j y)$. Conversely, to define $G : \mathscr{E}' \to \mathscr{E}$, consider $\varphi(x) \xrightarrow{\xi(x,y)} \psi(y)$ in $\mathscr{E}'$. The object $\varphi(x)$ is (up to equivalence in $\mathscr{T}_{\mathscr{E}}$) a formula of $\mathscr{L}_{\mathscr{E}}$; it has an interpretation $|\varphi(x)| \rightarrowtail i$ in $\mathscr{E}$ (where i is the type of x). We define G on the object $\varphi(x)$ by $G(\varphi(x)) = |\varphi(x)|$. Since $\xi(x,y)$ is an arrow in $\mathscr{E}'$, $\mathscr{T}_{\mathscr{E}} \vdash \forall_x^{\varphi} \exists!_y^{\psi} \xi(x,y)$, hence $\mathscr{E}$ satisfies this formula and by functional completeness of $\mathscr{T}$ (Lemma 10), this defines an arrow $|\varphi| \xrightarrow{f} |\psi|$ (denoted by $|[\xi]|$ in Lemma 12). We define $G(\xi(x,y))$ to be precisely that arrow. It is easy to show that G is a functor, but strictly speaking G is well defined only if there is in $\mathscr{E}$ a canonical choice of a mono representing a subobject. Lemma 6 shows that we produce only such toposes (and the equivalence we establish is in fact between theories and toposes with canonical choice).

Given this precaution, it is clear that $G \circ F = \mathrm{id}_{\mathscr{E}}$. On the other hand, $F \circ G \simeq \mathrm{id}_{\mathscr{E}'}$, since $F(G(\varphi(x))) \equiv (x =_{|\varphi|} x) \simeq \varphi(x)$ because, by Lemma 11, $\mathscr{E}' \models \forall x^{\varphi}\, \exists! y\, (my = x)$, with x of type i, y of type $|\varphi|$ and m the injection $|\varphi| \overset{m}{\rightarrowtail} i$.

(b) If $F : \mathscr{E}_1 \to \mathscr{E}_2$ is a left-exact functor, one easily verifies the naturality conditions embodied in the following two commuting squares:

$$\begin{tikzcd}
\mathscr{E}_1 \arrow[r, shift left, "F_1"] \arrow[d, "F"'] & \mathscr{E}_1' \arrow[l, shift left, "G_1"] \arrow[d, "\Phi(\Psi(F))"] \\
\mathscr{E}_2 \arrow[r, shift left, "F_2"] & \mathscr{E}_2' \arrow[l, shift left, "G_2"]
\end{tikzcd}$$

5.2. We now consider the composition $\Psi \circ \Phi$:

$$\mathbf{Th} \xrightarrow{\Phi} \mathbf{Topos} \xrightarrow{\Psi} \mathbf{Th}.$$

Let $\mathscr{T}$ be a theory and $\mathscr{T}' = \Psi(\Phi(\mathscr{T}))$. We construct logical translations

$$\begin{tikzcd}
\mathscr{T} \arrow[r, shift left, "{(\eta, e)}"] & \mathscr{T}' \arrow[l, shift left, "{(\theta, u)}"]
\end{tikzcd}$$

which are quasi-inverses one to another. More precisely, $(\theta, u) \circ (\eta, e)$ is the identical translation on $\mathscr{T}$, while $(\eta, e)\circ(\theta, u)$ is isomorphic to the identity on $\mathscr{T}'$: there is an isomorphic comparison γ from $(\eta, e)\circ(\theta, u)$ to the identity.

Before proceeding to these constructions, we recall how $\mathscr{T}'$ is constructed from $\mathscr{T}$: the types of $\mathscr{T}'$ are (equivalence classes of) formulas $\varphi(x)$ of $\mathscr{T}$; the operation symbols $\mathrm{Op}'_{\varphi(x),\psi(y)}$ are the arrows of $\mathscr{E}_{\mathscr{T}}$ from $\varphi(x)$ to $\psi(y)$; the relation symbols $\mathrm{Rel}'_{\varphi(x)}$ are the subobjects of $\varphi(x)$ in $\mathscr{E}_{\mathscr{T}}$; the product of types $\varphi(x)$ and $\psi(y)$ in $\mathscr{T}'$ is (essentially) conjunction in $\mathscr{T}$, while the exponential $\Omega^{\varphi(x)}$ is the $\mathscr{T}$-formula $\forall x\in X\,\varphi(x)$; finally, Γ' is the theory of $\mathscr{E}_{\mathscr{T}}$.

5.3. We describe here the logical translation $(\eta, e):\mathscr{T}\to\mathscr{T}'$.

(a) To type i, η associates $(x=_i x)$, i.e. 'true' in the variable x of type i. Clearly η respects not only 1 and products but also exponentials: $\eta(\Omega^i)$ is $(X=_{\Omega^i}X)$, which is 'true' in X, but $\Omega^{\eta(i)}$ is the power-object of $(x=_i x)$ in $\mathscr{E}_{\mathscr{T}}$, which is $\forall x\in X\,(x=_i x)$, which is again 'true' in X.

(b) To type i, e associates the $\mathscr{T}'$-formula $z=_{(x=_i x)}z$ where z is a variable of $\mathscr{T}'$ of type $x=_i x$ (this is indeed a type of $\mathscr{T}'$!). It is clear that e preserves the structure of types: e.g. $e_{\Omega^i}(X)$ is 'true' in X and so is $\forall x\in X\,e_i(x)$.

(c) To describe η on formulas we rely on our remarks on logical translations at the end of 2.2. We define $\eta:\mathrm{Rel}_i\to\mathrm{Rel}_{\eta(i)}$ and $\eta:\mathrm{Op}_{i,j}\to\mathrm{Op}'_{\eta(i),\eta(j)}$ by

$$\eta(r)\equiv(rx\rightarrowtail x=_i x)$$

and $\eta(f)\equiv(x=_i x\xrightarrow{fx=y}y=_j y)$ respectively.

Proposition 17. *(η, e) is a logical translation.*

The proof relies essentially on the following lemma which is proved by induction on expressions of $\mathscr{L}$:

Lemma 18. *For all formulas φ of $\mathscr{L}$, one has in $\mathscr{E}_{\mathscr{T}}$ the equality*

$$|\eta(\varphi(x))(x')|_{\mathscr{E}_{\mathscr{T}}}=\varphi(z)\overset{z=x}{\rightarrowtail}\eta i\quad(i.e.\ x=_i x)$$

and for all terms t the identity in $\mathscr{E}_{\mathscr{T}}$ of the arrows

$$x=_i x\underset{t(x)=y}{\overset{|\eta(t(x))(x')|}{\rightrightarrows}}y=_j y.$$

(x' is a variable of type $\eta(i)$.)

A word of comment is necessary here: the induction shows that in $\mathscr{E}_{\mathscr{T}}$ one has not simply an isomorphism but an equality between $\varphi(x)$ and $|\eta(\varphi(x))(x')|$: more exactly, $\varphi(x)$ and $|\eta(\varphi(x))(x')|$ are formulas which define the same object of $\mathscr{E}_{\mathscr{T}}$. Consider for example the inductive step of an atomic formula $\mathrm{rt}(x)$. By the very definition of interpretation, $|\mathrm{rt}(x)y|$ is 'the' pullback in $\mathscr{E}_{\mathscr{T}}$:

$$\begin{tikzcd} |(\mathrm{rt}(x))(y)| \arrow[r] \arrow[d, tail] & rx \arrow[d, tail, "x=z"] \\ x'=_i x' \arrow[r, "|t(x')(y)|"'] & z=_j z \end{tikzcd}$$

By inductive hypothesis $|t(x')(y)|$ is the equivalence class of $t(x')=z$. But in $\mathscr{E}_{\mathscr{T}}$ this pullback is canonically given by the formula $x'=x' \wedge rx \wedge \exists z\ (x=z \wedge t(x')=z)$ which is equivalent modulo $\mathscr{T}$ to $\mathrm{rt}(x')$ and hence defines the same object of $\mathscr{E}_{\mathscr{T}}$. The scrupulous reader might extend here to pullbacks the remarks we made on canonical subobjects.

5.4. In this paragraph we define the logical translation $(\theta, u) : \mathscr{T}' \to \mathscr{T}$.

(a) To type $\varphi(x)$ of $\mathscr{T}'$, θ simply associates the type i of the variable x. It is trivial that θ preserves the whole structure of types. Note that we loose here most of the information given in $\varphi(x)$: this information will be carried by the second component u of the translation and this is the keypoint where 'universes' find their justification.

(b) To type $\varphi(x)$ of $\mathscr{T}'$, u associates $\varphi(x)$, now viewed as a formula of $\mathscr{L}$. It is trivial that u preserves the whole structure of types.

(c) To describe θ on formulas we again rely on our remarks at the end of 2.2. We first define $\theta : \mathrm{Rel}_{i'} \to \mathrm{Form}$ where i' is an object $\psi(y)$ of $\mathscr{E}_{\mathscr{T}}$. Let $r' \equiv (\varphi(x) \overset{\xi(x,y)}{\rightarrowtail} \psi(y))$ be an element of $\mathrm{Rel}_{i'}$, i.e. a subobject of $\psi(y)$ in $\mathscr{E}_{\mathscr{T}}$. By Lemma 6, there exists a formula $\varrho(z)$, unique up to equivalence in $\mathscr{T}$, such that $\varrho(z) \overset{z=y}{\rightarrowtail} \psi(y)$ describes the same subobject as ξ. We let $\theta(r')$ be $\varrho(z)$. We now define θ on $\mathrm{Op}_{i',j'}$. Let $f' \in \mathrm{Op}_{i',j'}$; f' is an arrow in $\mathscr{E}_{\mathscr{T}}$, say $\varphi(x) \xrightarrow{\xi(x,y)} \psi(y)$; to make our definition independent of the choice of the formula $\xi(x,y)$ representing f' it suffices to define $\theta(f')$ to be $\varphi(x) \wedge \psi(y) \wedge \xi(x,y)$. The functionality condition relative to universes described by u is then simply the functionality of ξ relative to φ and ψ.

Proposition 19. *(θ, u) is a logical translation.*

The proof is based on the following lemma which is proved by induction on expressions of $\mathscr{L}$. Of course, comments analogous to those following Lemma 18 still apply.

Lemma 20. *For all formulas φ' of $\mathscr{L}'$, one has in $\mathscr{E}_{\mathscr{T}}$ the equality*

$$|\varphi'(x')|_{\mathscr{E}_{\mathscr{T}}} = \theta(\varphi'(x'))(z) \overset{z=_j y}{\rightarrowtail} i' \quad (\equiv \psi(y))$$

(*with x' of type i' and z of type $\theta(\psi(y))$, which is the type of y*), *and for all terms*

t' the identity

$$i' \equiv \varphi(x) \underset{\theta(t'(x'))(x,y)}{\overset{|t'(x')|}{\underset{\longrightarrow}{\overset{\longrightarrow}{\|}}}} \psi(y) \equiv j'$$

(with x' of type i' and $t'(x')$ of type j').

5.5. Let us compute $(\theta, u) \circ (\eta, e)$.

Proposition 21. *$(\theta, u) \circ (\eta, e)$ is the identical translation on $\mathcal{T}$.*

Proof. Consider for example a formula $\varphi(x)$ of $\mathcal{L}$ and $\theta(\eta(\varphi(x))(y))(z)$, where x is a variable of $\mathcal{L}$ of type i, y a variable of $\mathcal{L}'$ of type $\eta(i)$ (i.e. $x =_i x$) and z a variable of $\mathcal{L}$ of type $\theta(\eta(i))$ (i.e. $\theta(x =_i x)$, i.e. i). By Lemma 18 on η, it follows that

$$|\eta(\varphi(x))(y)| = \varphi(z) \overset{z=x}{\rightarrowtail} (x =_i x)$$

and by Lemma 20 on θ,

$$\theta(\eta(\varphi(x))(y))(z) = |\eta(\varphi(x))(y)|,$$

hence

$$\theta(\eta(\varphi(x))(y))(z) = \varphi(z).$$

5.6. Finally, let us compute $(\theta', u') = (\eta, e) \circ (\theta, u)$. Note that θ' is far from being the identity on types. Indeed, for type $\varphi(x)$ of $\mathcal{L}'$, $\theta'(\varphi(x))$ is $(x =_i x)$ (if x is of type i). On the other hand,

$$\begin{aligned} u'(\varphi(x)) &\equiv \eta(u(\varphi(x))) \wedge e(\theta(\varphi(x))) \\ &\equiv \eta(\varphi(x)) \wedge z =_{(x =_i x)} z, \end{aligned}$$

which is equivalent in $\mathcal{T}'$ to $\eta(\varphi(x))(z)$, which by Lemma 18, has as interpretation in $\mathcal{E}_{\mathcal{T}}$ the formula $\varphi(x)$ itself. What one has to do is to compare (θ', u') with the identity $(\mathrm{id}_1, \mathrm{id}_2)$ of $\mathcal{T}'$: $\mathrm{id}_1(i') = i'$, $\mathrm{id}_2(i') = (z =_{i'} z)$ and $\mathrm{id}_1(\varphi'(x')) = \varphi'(x')$.

Here is the description of an isomorphic comparison

$$\gamma : (\theta', u') \Rightarrow (\mathrm{id}_1, \mathrm{id}_2).$$

We rely here on our remarks at the end of Section 2. Let i' be a type of $\mathcal{L}'$, i.e. a formula $\varphi(x)$ of $\mathcal{L}$. We know that $u'(i')$ is $\eta(\varphi(x))(z)$ while $\mathrm{id}_2(i')$ is $z' =_{\varphi(x)} z'$. It is tempting to say that $\gamma(z, z')$ is $z = z'$, but this does not make sense since the types of z and z' are not the same: z is a variable of type $(x =_i x)$, while z' is of type $\varphi(x)$. To find the correct γ, look at the situation in $\mathcal{E}_{\mathcal{T}}$:

$$|z' =_{\varphi(x)} z'| \simeq \varphi(x)$$

and this is a subobject in canonical form of $x =_i x$. In other words, there is in $\mathcal{E}_{\mathcal{T}}$

a mono

$$\varphi(x) \xrightarrow{x=y} y =_i y$$

which in turn becomes an operation symbol in $\mathscr{L}'$, more exactly, an element f' of $\mathrm{Op}'_{i',j'}$, where i' is $\varphi(x)$ and j' is $y=_i y$. To this f' corresponds in $\mathscr{L}'$ the functional relation $z =_{(x=_i x)} f'z'$ where z is a variable of type $x=_i x$ and z' a variable of type $\varphi(x)$ and consequently $f'z'$ a term of type $x=_i x$. We define thus

$$\gamma_{\varphi(x)}(z, z') = (z =_{(x=_i x)} f'z').$$

It is not surprising that this defines an isomorphic $\gamma : (\theta', u') \Rightarrow (\mathrm{id}_1, \mathrm{id}_2)$. Observe for example that the condition

$$\mathscr{T}' \vdash \forall z^{u'(\varphi(x))}\, \exists! z'\, (z =_{(x=_i x)} f'z') \wedge \forall z'\, \exists! z^{u'(\varphi(x))}\, (z =_{(x=_i x)} f'z')$$

essentially means that f' is in $\mathscr{E}_{\mathscr{T}}$ an iso from φ to φ! Technically, it suffices to apply Lemma 11 on relativized formulas. We omit the other verifications to summarize our discussion as follows:

Proposition 22. *$(\eta, e) \circ (\theta, u)$ is isomorphic to the identity.*

We have thus developed the essentials of the analysis leading to

Theorem 23. *The 2-category of toposes with left-exact functors as 1-arrows and natural transformations as 2-arrows is equivalent to the 2-category of theories with translations as 1-arrows and comparisons as 2-arrows. In this equivalence, functors preserving finite unions (resp. functors preserving images, logical functors) correspond to translations preserving disjunctions (resp. translations preserving the existential quantifiers, logical translations).*

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